Bounds for pairs in partitions of graphs

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Abstract

In this paper we study the following problem of Bollobás and Scott: What is the smallest $f(k,m)$ such that for any integer $k \geq 2$ and any graph G with m edges, there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that for $1 \leq i \neq j \leq k$, $e(V_i \cup V_j) \leq f(k,m)$? We show that $f(k,m) < 1.6m/k + o(m)$, and $f(k,m) < 1.5m/k + o(m)$ for $k \geq 23$. (While the graph $K_{1,n}$ shows that $f(k,m) \geq m/(k-1)$, which is $1.5m/k$ when $k = 3$.) We also show that $f(4,m) \leq m/3 + o(m)$ and $f(5,m) \leq 4m/15 + o(m)$, providing evidence to a conjecture of Bollobás and Scott. For dense graphs, we improve the bound to $4m/k^2 + o(m)$, which, for large graphs, answers in the affirmative a related question of Bollobás and Scott.

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1 Introduction

For a graph G, we use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. We use $\delta(G)$ to denote the minimum degree of G. For subsets S, T of $V(G)$, we use $e(S,T)$ to denote the number of edges of G with one end in S and the other in T; $e(S)$ to denote the number of edges with both ends in S; and $d(S)$ to denote the number of edges with at least one end in S.

Classical graph partition problems often ask for partitions of a graph that optimize a single quantity. For example, the *Maximum Bipartite Subgraph Problem* asks for a partition V_1, V_2 of the vertices of a graph that maximizes $e(V_1, V_2)$. This problem is NP-hard, see [11]. However, it is easy to prove that any graph with m edges has a partition V_1, V_2 with $e(V_1, V_2) \geq m/2$. Edwards [8,9] improved this lower bound to $m/2+\frac{1}{4}(\sqrt{2m+1/4}-1/2)$, which is best possible for complete graphs K_{2n+1}.

A different type of partition problems ask for a partition of a given graph that optimizes several quantities simultaneously. Such problems are called *Judicious Partition Problems* by Bollobás and Scott [3]. The *Bottleneck Bipartition Problem*, raised by Entringer (see, for example, [13, 15]) is a judicious partition problem: Find a partition V_1, V_2 of the vertex set of a graph G that minimizes $\max\{e(V_1), e(V_2)\}$. Shahrokhi and Székely [16] showed that this problem is also NP-hard. Porter [13] proved that any graph with m edges has a partition V_1, V_2 with $e(V_i) \leq m/4 + O(\sqrt{m})$, establishing a conjecture of Erdös. (A matrix version of this Erdös conjecture was formulated by Entringer, and was solved by Porter and Székely [14].) Bollobás and Scott [5] improved this to $e(V_i) \leq m/4 + \frac{1}{8}(\sqrt{2m+1/4} - 1/2)$, and showed that K_{2n+1} are the only extremal graphs. We note that in [1] a connection is given between the Maximum Bipartite Subgraph Problem and the Bottleneck Bipartition Problem.

Bollobás and Scott [5] proved that for any integer $k \geq 1$ and any graph G of size m, $V(G)$ can be partitioned into $V_1, \ldots, V_k$ such that $e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2)$ for $i \in \{1, 2, \ldots, k\}$. The complete graphs of order $kn+1$ are the only extremal graphs (modulo isolated vertices).

In this paper we study the following judicious partition problem of Bollobás and Scott [7].

Problem 1.1 *What is the smallest $f(k,m)$ such that for any integer $k \geq 2$ and any graph G with m edges, there is a partition $V(G) = \bigcup_{i=1}^{k} V_i$ such that for $1 \leq i \neq j \leq k$, $e(V_i \cup V_j) \leq f(k,m)$?*

Note that the case $k = 2$ for Problem 1.1 is trivial. For $k = 3$, we note that for each permutation ijk of $\{1, 2, 3\}$, $d(V_i) = m - e(V_j \cup V_k)$; so Problem 1.1 asks for a lower bound on $\min\{d(V_i) : i = 1, 2, 3\}$ which is studied in [12]. For $k \geq 4$, bounding $\max\{e(V_i \cup V_j) : 1 \leq i \neq j \leq k\}$ is much more difficult than bounding $\max\{e(V_i) : 1 \leq i \leq k\}$: In the former case one needs to bound $\binom{k}{2}$ quantities resulted from a k-partition, while in the latter case one only needs to bound k quantities.

In Section 2, we use probabilistic method to show that $f(k,m) < 1.6m/k + o(m)$, and that $f(k,m) < 1.5m/k + o(m)$ for $k \geq 23$.

The following example shows that $f(k,m) \geq m/(k-1)$, which is close to $1.6m/k$ when $k = 3$. For $k \geq 3$, take the graph $K_{1,n}$ with $n \geq k-1$, and let x be the vertex of degree n. Let $V_1, \ldots, V_k$ be a k-partition of $V(G)$, with $x \in V_1$. Without loss of generality, we may assume that $|V_2| \geq (n+1-|V_1|)/(k-1)$. Now $e(V_1 \cup V_2) \geq (n+1-|V_1|)/(k-1) + (|V_1|-1) =$

$(n+(k-2)(|V_1|-1))/(k-1) \geq n/(k-1) = m/(k-1)$, where m is the number of edges in $K_{1,n}$.

On the other hand, the complete graph K_{k+2} has $m = \binom{k+2}{2}$ edges, and any k-partition $V_1, \dots, V_k$ of K_{k+2} has two sets, say V_1, V_2, such that $|V_1 \cup V_2| = 4$. So $e(V_1 \cup V_2) = 6 = \frac{12m}{(k+2)(k+1)}$. This shows that $f(k,m) \geq \frac{12m}{(k+2)(k+1)}$. For general complete graphs K_n, a simple counting shows that for any k-partition $V_1, \dots, V_k$ of K_n, $k \geq 2$, there exist V_i, V_j such that $|V_i| + |V_j| \geq 2n/k$; and hence $e(V_i \cup V_j) \geq \binom{2n/k}{2}$. From this, we can deduce that $f(k,m) \geq 4m/k^2 + O(n)$, and this bound is achieved by taking a balanced k-partition of $V(K_n)$ (i.e., any two partition sets differ in size by at most one).

Note that $K_{1,n}$ is sparse, i.e. the number of edges is $O(n)$. The consideration of $K_{1,n}$ and K_{k+2} led Bollobás and Scott [7] to the following conjecture.

Conjecture 1.2 $f(k,m) \leq \frac{12m}{(k+2)(k+1)} + O(n)$.

The case $k = 2$ for Conjecture 1.2 is trivial (as the bound becomes $m + O(n)$).

For $k = 3$, Conjecture 1.2 is equivalent to the following problem: Find a partition $V(G) = V_1 \cup V_2 \cup V_3$ so that $d(V_i) \geq 2m/5 + O(n)$. It is shown in [12] that if G is a graph with m edges then there is a partition $V_1, \dots, V_k$ of $V(G)$ such that $d(V_i) \geq m/(k-1) + O(m^{4/5})$ (establishing a conjecture of Bollobás and Scott [6, 7], for large graphs). This result implies $f(3,m) < m/2 + o(m^{4/5})$; so Conjecture 1.2 holds for $k = 3$.

In Section 3, we prove the bound $4m/k^2 + o(m)$ for dense graphs, which implies that Conjecture 1.2 holds for dense graphs. As a consequence, we establish the following conjecture of Bollobás and Scott [7] for large graphs.

Conjecture 1.3 *For each $k \geq 2$ there is a constant $c_k > 0$ such that if G is a graph with m edges, n vertices, and $\delta(G) \geq c_k n$, then there is a partition $V_1, \dots, V_k$ of $V(G)$ such that for $1 \leq i \neq j \leq k$,*

$$e(V_i, V_j) \leq \frac{12m}{(k+2)(k+1)}.$$

In Section 4, we show $f(4,m) \leq m/3 + o(m)$ and $f(5,m) \leq 4m/15 + 0(m)$, which implies Conjecture 1.2 for $k = 4$ and $k = 5$.

In Section 5, we study partitions $V_1, \dots, V_k$ of graphs that optimize both $\max\{e(V_i) : 1 \leq i \leq k\}$ and $\max\{e(V_i \cup V_j) : 1 \leq i \neq j \leq k\}$. Bollobás and Scott [7] asked whether it is possible to find a partition $V_1, \dots, V_k$ such that $e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}(\sqrt{2m+1/4} - 1/2)$ for $1 \leq i \leq k$, and $e(V_i \cup V_j) \leq \frac{12m}{(k+2)(k+1)} + O(n)$ for $1 \leq i \neq j \leq k$. We show that for $k = 3$ and $k = 4$ one can find a partition satisfying these bounds asymptotically.

2 A bound for k-partitions

In this section, we prove a bound on $f(k,m)$ in Problem 1.1. First, we state the Azuma-Hoeffding inequality [2, 10], which will be used to bound deviations. We use the version given in [4].

Lemma 2.1 *Let $Z_1, \dots, Z_n$ be independent random variables taking values in $\{1, \dots, k\}$, let $Z := (Z_1, \dots, Z_n)$, and let $f : \{1, \dots, k\}^n \to \mathbf{N}$ such that $|f(Y) - f(Y')| \le c_i$ for any $Y, Y' \in \{1, \dots, k\}^n$ that differ only in the i^{th} coordinate. Then for any $z > 0$,*

$$\mathbb{P}\left(f(Z) \ge E(f(Z)) + z\right) \le \exp\left(\frac{-z^2}{2\sum_{i=1}^k c_i^2}\right),$$

$$\mathbb{P}\left(f(Z) \le E(f(Z)) - z\right) \le \exp\left(\frac{-z^2}{2\sum_{i=1}^k c_i^2}\right).$$

We need a simple lemma which will also be used in Section 4 for finding probabilities when finding 4-partitions.

Lemma 2.2 *Let $a_j \ge 0$ for $j \in \{1, 2, 3, 4\}$ such that $\alpha := \sum_{j=1}^4 a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \le i \ne j \le 4$. Then there exist $p_i \in [0, 1/2]$, $1 \le i \le 4$, such that $\sum_{i=1}^4 p_i = 1$ and, for $1 \le i \ne j \le 4$, $f_{ij}(p_i, p_j) \le \alpha/3$.*

Proof. First, assume $a_i \le \alpha/2$ for all $1 \le i \le 4$. Then $p_i := 1/2 - a_i/\alpha \in [0, \frac{1}{2}]$, and

$$f_{ij}(p_i, p_j) = (a_i + a_j)\left(1 - \frac{a_i + a_j}{\alpha}\right) = -\frac{1}{\alpha}(a_i + a_j - \frac{\alpha}{2})^2 + \frac{\alpha}{4} \le \frac{\alpha}{4}.$$

So we may assume without loss of generality that $a_4 > \alpha/2$. Then $a_i + a_j \le \alpha/2$ for all $1 \le i \ne j \le 3$. Let $p_1 = p_2 = p_3 = 1/3$ and $p_4 = 0$. Then for $1 \le i \le 3$, $f_{i4} = (a_i + a_4)/3 \le \alpha/3$; and for $1 \le i \ne j \le 3$, $f_{ij} = (a_i + a_j)(2/3) \le (\alpha/2)(2/3) = \alpha/3$. ∎

Remark. From the proof of Lemma 2.2, we may choose $p_i = 0$ when $a_i > \alpha/2$, and $p_i \le \max\{1/2 - a_i/\alpha, 1/3\}$ when $a_i \le \alpha/2$.

We need another lemma.

Lemma 2.3 *Let $h_4 = 1/3$. There exist t_k, h_k for $k \ge 5$ such that*

$$\begin{aligned} h_k &= \frac{2 - 2t_k}{k - 2t_k}, \text{ and} \\ \frac{2 - 2t_k}{k - 2t_k} &= \frac{k-3}{k} h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right) 2t_k. \end{aligned}$$

Moreover, $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$.

Proof. We first show that there exist $t_k \in (0, 1/2)$ and $h_k \in (1/(k-1), 2/k)$, $k \ge 5$, such that

$$\begin{aligned} h_k &= \frac{2 - 2t_k}{k - 2t_k}, \text{ and} \\ \frac{2 - 2t_k}{k - 2t_k} &= \frac{k-3}{k} h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right) 2t_k. \end{aligned}$$

Suppose $k \ge 5$. Let

$$f_k(t) = \frac{2 - 2t}{k - 2t}$$

and

$$g_k(t) = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t.$$

It is easy to see that $f_k(t)$ is decreasing, and $g_k(t)$ is increasing. Now assume that $\frac{1}{k-1} \leq h_{k-1} < \frac{2}{k-1}$ for some $k \geq 5$. Note that

$$g_k(0) = \frac{k-3}{k}h_{k-1} < \frac{k-3}{k}\frac{2}{k-1} < \frac{2}{k} = f_k(0),$$

and

$$g_k(1/2) = \frac{k-2}{k}h_{k-1} + \frac{4}{k(k-1)} \geq \frac{k-2}{k(k-1)} + \frac{4}{k(k-1)} > \frac{1}{k-1} = f_k(1/2).$$

Therefore, since $f_k(t)$ is decreasing and $g_k(t)$ is increasing, there exists $t_k \in (0, 1/2)$, for each $k \geq 5$, such that $f_k(t_k) = g_k(t_k)$. Let $h_k := f_k(t_k) = \frac{2-2t_k}{k-2t_k}$. Then since $t_k \in (0, 1/2)$, $1/(k-1) < h_k < 2/k$ for $k \geq 5$.

Next, we show that $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \geq 23$. Let $h_k = c_k/k$, and it suffices to show $c_k < 1.6$, and $c_k < 1.5 = 3/2$ for $k \geq 23$. Since $h_k \in (1/(k-1), 2/k)$, $c_k \in (1, 2)$. Note that

$$c_k = \frac{2-2t_k}{k-2t_k}k = (k-3)h_{k-1} + \left(h_{k-1} + \frac{4}{k-1}\right)2t_k = \frac{k-3}{k-1}c_{k-1} + \frac{4+c_{k-1}}{k-1}2t_k.$$

From $c_k = \frac{2-2t_k}{k-2t_k}k$ we deduce $t_k = \frac{2k-kc_k}{2k-2c_k}$; and so

$$c_k = \frac{k-3}{k-1}c_{k-1} + \frac{(4+c_{k-1})(2k-kc_k)}{(k-1)(k-c_k)}.$$

With $h_4 = 1/3$ (and hence $c_4 = 4/3$) and using *MATLAB*, we have $c_k < 1.6$ for $k = 5, \ldots, 22$, and $c_{23} \approx 1.4962 < 3/2$. Now assume $k \geq 24$ and $c_{k-1} < 3/2$. Then

$$c_k < \frac{k-3}{k-1}\left(\frac{3}{2}\right) + \frac{(4+3/2)(2k-kc_k)}{(k-1)(k-c_k)},$$

and so

$$2(k-1)c_k < 3(k-3) + 11(2-c_k) + 11(2-c_k)c_k/(k-c_k).$$

Hence, since $c_k \in (1, 2)$,

$$(2k+9)c_k < 3k+13 + \frac{11(2-c_k)c_k}{k-c_k} = 3k+13 + \frac{11(1-(1-c_k)^2)}{k-c_k} < 3k+13+11/(k-2).$$

Therefore,

$$c_k < \frac{3k+13}{2k+9} + \frac{11}{(2k+9)(k-2)} \leq 3/2.$$

The last inequality holds since we assume $k \geq 24$. ∎

We can now prove the main lemma for k-partitions.

Lemma 2.4 *Let $k \geq 4$ be an integer, let $a_j \geq 0$ for $j \in \{1, \dots, k\}$ such that $\alpha := \sum_{j=1}^{k} a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq k$. Let h_k be defined as in Lemma 2.3. Then there exist $p_i \in [0, 2/k]$, $1 \leq i \leq k$, such that $\sum_{i=1}^{k} p_i = 1$ and, for $1 \leq i \neq j \leq k$, $f_{ij}(p_i, p_j) \leq 1.6\alpha/k$, and $f_{ij}(p_i, p_j) \leq 1.5\alpha/k$ for $k \geq 23$.*

Proof. We apply induction on k; the case $k = 4$ follows from Lemma 2.2 (as $h_4 = 1/3$). Suppose $k \geq 5$.

First, assume that there exists some $l \in \{1, \dots, k\}$ such that $a_l \geq t\alpha$, say $l = k$. Let $p_i = x$ for $1 \leq i < k$, with $0 \leq x \leq \frac{1}{k-1}$, and let $p_k = 1-(k-1)x$. Then $\sum_{i=1}^{k} p_i = 1$; for $1 \leq i \leq k-1$,

$$f_{ik}(p_i, p_k) \leq (1 - (k-2)x)\alpha;$$

and for $1 \leq i \neq j \leq k-1$,

$$f_{ij}(p_i, p_j) \leq 2x(a_i + a_j) \leq 2x(\alpha - a_k) \leq (1-t)2x\alpha.$$

We wish to minimize $\max\{1 - (k-2)x, (1-t)2x\}$. Setting $1 - (k-2)x = (1-t)2x$, we have

$$x = \frac{1}{k - 2t}$$

and, for $1 \leq i \neq j \leq k$,

$$f_{ij}(p_i, p_j) \leq \frac{2 - 2t}{k - 2t}\alpha.$$

Since $0 \leq x \leq \frac{1}{k-1}$ and $x = 1/(k-2t)$, we have $0 \leq t \leq \frac{1}{2}$.

Now let us assume that $a_i \leq t\alpha$ for all $1 \leq i \leq k$. By induction hypothesis, for any $l \in \{1, \dots, k\}$ there exist $p_i^l \in [0, 2/(k-1)]$, $i \in \{1, \dots, k\} \setminus \{l\}$, such that $\sum_{i \in \{1,\dots,k\}\setminus\{l\}} p_i^l = 1$ and for any $\{i, j\} \subseteq \{1, \dots, k\} \setminus \{l\}$,

$$(a_i + a_j)(p_i^l + p_j^l) \leq h_{k-1}(\alpha - a_l).$$

For $1 \leq i \leq k$, let

$$p_i = \frac{1}{k} \sum_{l \in \{1,\dots,k\}\setminus\{i\}} p_i^l.$$

Since $p_i^l \leq 2/(k-1)$ for $i \in \{1, \dots, k\} \setminus \{l\}$, we have $p_i \in [0, 2/k]$ for $1 \leq i \leq k$. Also,

$$\sum_{i=1}^{k} p_i = \frac{1}{k} \sum_{i=1}^{k} \sum_{l \in \{1,\dots,k\}\setminus\{i\}} p_i^l = \frac{1}{k} \sum_{l=1}^{k} \sum_{i \in \{1,\dots,k\}\setminus\{l\}} p_i^l = \frac{1}{k} \sum_{l=1}^{k} 1 = 1.$$

Moreover, for $1 \le i \ne j \le k$,

$$\begin{aligned}
f_{ij}(p_i, p_j) &= (a_i + a_j)(p_i + p_j) \\
&= \frac{1}{k}(a_i + a_j)\left(\sum_{l\in\{1,\dots,k\}\setminus\{i\}} p_i^l + \sum_{l\in\{1,\dots,k\}\setminus\{j\}} p_j^l\right) \\
&= \frac{1}{k}\left(\sum_{l\in\{1,\dots,k\}\setminus\{i,j\}} (a_i + a_j)(p_i^l + p_j^l)\right) + \frac{1}{k}(a_i + a_j)(p_i^j + p_j^i) \\
&\le \frac{h_{k-1}}{k}\sum_{l\in\{1,\dots,k\}\setminus\{i,j\}} (\alpha - a_l) + \frac{1}{k}(a_i + a_j)(p_i^j + p_j^i) \\
&\le \frac{h_{k-1}}{k}\left((k-3)\alpha + a_i + a_j\right) + \frac{4}{k(k-1)}(a_i + a_j) \\
&\le \frac{k-3}{k}h_{k-1}\alpha + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t\alpha.
\end{aligned}$$

By Lemma 2.3 and since $h_4 = 1/3$, there exist t_k, h_k for $k \ge 5$ such that

$$h_k = \frac{2-2t_k}{k-2t_k} = \frac{k-3}{k}h_{k-1} + \left(\frac{h_{k-1}}{k} + \frac{4}{k(k-1)}\right)2t_k,$$

$h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$. This completes the proof of the lemma. ∎

Theorem 2.5 *Let $k \ge 4$ be an integer. Then $f(k,m) \le h_k m + O(m^{4/5})$, where $h_k < 1.6/k$, and $h_k < 1.5/k$ for $k \ge 23$.*

Proof. Let G be a graph with m edges, and we may assume that G is connected (as otherwise we simply consider individual components). Let $V(G) = \{v_1, \dots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \dots \ge d(v_n)$. Let $V_1 = \{v_1, \dots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$ and will be optimized later. Then $t < n$ since $m < n^2/2$. Moreover, $e(V_1) < t^2/2 \le \frac{1}{2}m^{2\alpha}$ and $d(v_{t+1}) \le 2m^{1-\alpha}$ (since $(t+1)d(v_{t+1}) \le \sum_{i=1}^{t+1} d(v_i) \le 2m$).

Label the vertices in $V_2 := V(G)\setminus V_1$ as $u_1, \dots, u_{n-t}$ such that $e(u_i, V_1 \cup \{u_1, \dots, u_{i-1}\}) > 0$ for $i = 1, \dots, n-t$. Note that this can be done since G is connected.

Fix a random k-partition $V_1 = \bigcup_{i=1}^k Y_i$, and assign each member of Y_i the color i, $1 \le i \le k$. Extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p_j^i, where $\sum_{j=1}^k p_j^i = 1$ and p_j^i will be determined later. Let Z_i denote the indicator random variable of the event of coloring u_i. Hence $Z_i = j$ iff u_i is assigned the color j.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \dots, n-t$, and let $G_0 = G[V_1]$. Let $X_j^0 = Y_j$ for $1 \le j \le k$, and $x_{jl}^0 = e(X_j^0 \cup X_l^0)$ for $1 \le j \ne l \le k$. For $i = 1, \dots, n-t$ and $1 \le j, l \le k$, define

$$\begin{aligned}
X_j^i &:= \{\text{vertices of } G_i \text{ with color } j\}, \\
x_{jl}^i &:= e(X_j^i \cup X_l^i), \\
\Delta x_{jl}^i &:= x_{jl}^i - x_{jl}^{i-1}, \\
b_j^i &:= e(u_i, X_j^{i-1}).
\end{aligned}$$

Note that b_j^i depends on $(Z_1,\dots,Z_{i-1})$ only. Hence for $1\le i\le n-t$ and $1\le j\ne l\le k$,

$$\mathbb{E}(\Delta x_{jl}^i|Z_1,\dots,Z_{i-1})=(b_j^i+b_l^i)(p_j^i+p_l^i),$$

and so

$$\mathbb{E}(\Delta x_{jl}^i)=(a_j^i+a_l^i)(p_j^i+p_l^i),$$

where here

$$a_j^i=\sum_{(Z_1,\dots,Z_{i-1})}\mathbb{P}(Z_1,\dots,Z_{i-1})b_j^i.$$

Since b_j^i is determined by $(Z_1,\dots,Z_{i-1})$, a_j^i is determined by p_j^s, $1\le j\le k$ and $1\le s\le i-1$. Note that $\sum\limits_{j=1}^k b_j^i=e(u_i,G_{i-1})>0$, and that $e(u_i,G_{i-1})$ is independent of $Z_1,\dots,Z_{n-t}$. Moreover,

$$\begin{aligned}\sum_{j=1}^k a_j^i&=\sum_{j=1}^k\sum_{(Z_1,\dots,Z_{i-1})}\mathbb{P}(Z_1,\dots,Z_{i-1})b_j^i\\&=\sum_{(Z_1,\dots,Z_{i-1})}\left(\mathbb{P}(Z_1,\dots,Z_{i-1})\sum_{j=1}^k b_j^i\right)\\&=\sum_{(Z_1,\dots,Z_{i-1})}\mathbb{P}(Z_1,\dots,Z_{i-1})e(u_i,G_{i-1})\\&=e(u_i,G_{i-1})\\&>0.\end{aligned}$$

So by Lemma 2.4, there exist $p_j^i\in[0,1]$, $1\le j\le k$, such that $\sum_{j=1}^k p_j^i=1$ and, for $1\le i\le n-t$ and $1\le j\ne l\le k$,

$$\mathbb{E}(\Delta x_{jl}^i)\le h_k\sum_{j=1}^k a_j^i=h_ke(u_i,G_{i-1}).$$

Note that p_j^i is determined by a_j^i, $1\le i\le k$; and hence p_j^i is recursively determined by p_j^s, $1\le j\le k$ and $1\le s\le i-1$. Also note that $m=e(G_0)+\sum_{i=1}^{n-t}e(u_i,G_{i-1})$. Now

$$\begin{aligned}\mathbb{E}(x_{jl}^{n-t})&=\sum_{i=1}^{n-t}\mathbb{E}(\Delta x_{jl}^i)+\mathbb{E}(x_{jl}^0)\\&\le h_k\sum_{i=1}^{n-t}e(u_i,G_{i-1})+x_{jl}^0\\&\le h_km+e(V_1)\\&\le h_km+\frac{1}{2}m^{2\alpha}.\end{aligned}$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $x_{jl} := x_{jl}^{n-t}$ by at most $d(u_i)$. So by Lemma 2.1,

$$\begin{aligned}
\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) &\leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)^2}\right) \\
&\leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)d(v_{t+1})}\right) \\
&< \exp\left(-\frac{z^2}{4m2m^{1-\alpha}}\right) \\
&\leq \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).
\end{aligned}$$

Let $z = (8\ln(k(k-1)/2))^{\frac{1}{2}}\, m^{1-\frac{\alpha}{2}}$. Then for $1 \leq j \neq l \leq k$,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) < \exp(-\ln(k(k-1)/2)) = \frac{2}{k(k-1)}.$$

So there exists a partition $V(G) = \bigcup_{i=1}^{k} X_i$ such that for $1 \leq j \neq l \leq k$,

$$e(X_j \cup X_l) \leq \mathbb{E}(x_{jl}) + z \leq h_k m + \frac{1}{2}m^{2\alpha} + z \leq h_k m + o(m),$$

where the $o(m)$ term in the expression is

$$\frac{1}{2}m^{2\alpha} + (8\ln(k(k-1)/2))^{\frac{1}{2}}\, m^{1-\frac{\alpha}{2}}.$$

Choosing $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

3 Dense graphs

We now prove Conjecture 1.2 for graphs with large minimum degree. The approach is similar to that for proving Theorem 2.5, but simpler because the large minimum degree condition helps to bound $e(V_1, V_2)$. Note that the term $4m/k^2$ in the theorem below is best possible (by simply taking a random k-partition).

Theorem 3.1 *Let $k \geq 2$ be an integer and let $\epsilon > 0$. If G is a graph with m edges and $\delta(G) \geq \epsilon n$, then there is a partition $X_1, \ldots, X_k$ of $V(G)$ such that for $1 \leq i \neq j \leq k$,*

$$e(X_i \cup X_j) \leq \frac{4}{k^2}m + \left(\sqrt{2/\epsilon} + \sqrt{8\ln\frac{k(k-1)}{2}}\right) m^{5/6}.$$

Proof. We may assume that G is connected (otherwise it suffices to consider individual components). Let $V(G) = \{v_1, ..., v_n\}$ such that $d(v_1) \geq d(v_2) \geq ... \geq d(v_n)$. Let $V_1 = \{v_1, ..., v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$. Then $t < n$, $e(V_1) \leq m^{2\alpha}/2$, and $d(v_{t+1}) \leq 2m^{1-\alpha}$. Let $V_2 = V(G)\backslash V_1 = \{u_1, ..., u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, ..., u_{i-1}\}) > 0$ for $i = 1, ..., n-t$.

Now assume $\delta(G) \geq \epsilon n$. Then $2m = \sum_{v \in V(G)} d(v) \geq \epsilon n^2$. So $n \leq \sqrt{2m/\epsilon}$. Thus,

$$e(V_1, V_2) + 2e(V_1) = \sum_{i=1}^{t} d(v_i) < tn \leq m^\alpha \sqrt{2m/\epsilon} = \sqrt{2/\epsilon} m^{1/2+\alpha}.$$

Fix a random partition $V_1 = Y_1 \cup Y_2 \cup ... \cup Y_k$ and, for each $i \in \{1, ..., k\}$, assign the color i to all vertices in Y_i. We extend this coloring to $V(G)$ by independently assigning the color j (for each $j \in \{1, ..., k\}$) to each vertex $u_i \in V_2$ with probability $1/k$. Let Z_i denote the indicator random variable of the event of coloring u_i.

Let X_i be the set of vertices of G with color i. Then $Y_i \subseteq X_i$ for $1 \leq i \leq k$; and for $1 \leq i \neq j \leq k$,

$$\begin{aligned}
\mathbb{E}(e(X_i \cup X_j)) &= \mathbb{E}(e((X_i \cup X_j) \cap V_2)) + \mathbb{E}(e((X_i \cup X_j) \cap V_2, Y_i \cup Y_j)) + e(Y_i \cup Y_j) \\
&\leq (2/k)^2 e(V_2) + e(V_1, V_2) + e(V_1) \\
&\leq \frac{4}{k^2} m + \sqrt{2/\epsilon} m^{1/2+\alpha}.
\end{aligned}$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $e(X_i \cup X_j)$ by at most $d(u_i)$. Then as in the proof of Theorem 2.5, we apply Lemma 2.1 to conclude that for any $1 \leq i \neq j \leq k$,

$$\mathbb{P}\left(e(X_i \cup X_j) > \mathbb{E}(e(X_i \cup X_j)) + z\right) \leq \exp\left(-\frac{z^2}{2\sum_{i=1}^{n-t} d(u_i)^2}\right) \leq \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).$$

Let $z = \sqrt{8 \ln(k(k-1)/2)} m^{1-\alpha/2}$. Then for $1 \leq i \neq j \leq k$,

$$\mathbb{P}\left(e(X_i \cup X_j) > \mathbb{E}(e(X_i \cup X_j)) + z\right) < \exp\left(-\ln \frac{k(k-1)}{2}\right) = \frac{2}{k(k-1)}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup ... \cup X_k$ such that, for $1 \leq i \neq j \leq k$,

$$\begin{aligned}
e(X_i \cup X_j) &\leq \frac{4}{k^2} m + \sqrt{2/\epsilon} m^{1/2+\alpha} + z \\
&\leq \frac{4}{k^2} m + \sqrt{2/\epsilon} m^{1/2+\alpha} + \sqrt{8 \ln(k(k-1)/2)} m^{1-\alpha/2}
\end{aligned}$$

Picking $\alpha = 1/3$ to minimize $\max\{1/2 + \alpha, 1 - \alpha/2\}$, we have the desired bound. ∎

As a corollary, Conjecture 1.3 holds for graphs with $\Omega(k^{12}(\ln k)^3)$ edges. Hence Conjecture 1.2 holds for all graphs G with $\delta(G) \geq \epsilon n$, for any fixed $k \geq 2$ and $\epsilon > 0$.

4 Bounds for 4-partitions and 5-partitions

In this section, we prove Conjecture 1.2 for 4-partitions and 5-partitions. We use Lemma 2.2 for 4-partitions. For 5-partitions, we need the following lemma.

Lemma 4.1 *Let $a_j \geq 0$ for $j \in \{1, \ldots, 5\}$ such that $\alpha := \sum_{j=1}^5 a_j > 0$, and let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq 5$. Then there exist $p_i \in [0, 2/5]$, $1 \leq i \leq 5$, such that $\sum_{i=1}^5 p_i = 1$ and, for $1 \leq i \neq j \leq 5$, $f_{ij}(p_i, p_j) \leq 4\alpha/15$.*

Proof. If there exists some $l \in \{1,\ldots,5\}$ such that $a_l \geq 5\alpha/11$, then $a_i + a_j \leq 6\alpha/11$ for $\{i,j\} \subseteq \{1,\ldots,5\} \setminus \{l\}$. Let $p_l = 1/45$ and let $p_i = 11/45$ for $i \in \{1,\ldots,5\} \setminus \{l\}$. Then for $i \in \{1,\ldots,5\} \setminus \{l\}$,

$$f_{il}(p_i,p_l) = (a_i + a_l)(p_i + p_l) \leq \alpha \left(\frac{11}{45} + \frac{1}{45}\right) = \frac{4}{15}\alpha;$$

and for $\{i,j\} \subseteq \{1,\ldots,5\} \setminus \{l\}$,

$$f_{ij} = (a_i + a_j)(p_i + p_j) \leq \frac{6\alpha}{11}\left(\frac{11}{45} + \frac{11}{45}\right) = \frac{4}{15}\alpha.$$

Therefore, we may assume that $a_i < 5\alpha/11$ for all $1 \leq i \leq 5$. By Lemma 2.2, for any $1 \leq l \leq 5$ there exist $p_i^l \in [0,1/2]$, $i \in \{1,\ldots,5\} \setminus \{l\}$, such that $\sum_{i\in\{1,\ldots,5\}\setminus\{l\}} p_i^l = 1$ and, for $\{i,j\} \subseteq \{1,\ldots,5\} \setminus \{l\}$,

$$(a_i + a_j)(p_i^l + p_j^l) \leq \frac{1}{3}(\alpha - a_l).$$

Indeed, by the remark following Lemma 2.2, we may choose p_i^l, $i \in \{1,\ldots,5\} \setminus \{l\}$, such that $p_i^l = 0$ when $a_i > (\alpha - a_l)/2$, and $p_i^l \leq \max\{1/2 - a_i/(\alpha - a_l), 1/3\}$ when $a_i \leq (\alpha - a_l)/2$.

For $1 \leq i \leq 5$, let $p_i = \frac{1}{5}\sum_{l\in\{1,\ldots,5\}\setminus\{i\}} p_i^l$. Then $p_i \in [0, 2/5]$, and

$$\sum_{i=1}^{5} p_i = \frac{1}{5}\sum_{i=1}^{5}\sum_{l\in\{1,\ldots,5\}\setminus\{i\}} p_i^l = \frac{1}{5}\sum_{l=1}^{5}\sum_{i\in\{1,\ldots,5\}\setminus\{l\}} p_i^l = \frac{1}{5}\sum_{l=1}^{5} 1 = 1.$$

So for $1 \leq i \neq j \leq 5$,

$$\begin{aligned}
f_{ij}(p_i,p_j) &= (a_i + a_j)(p_i + p_j) \\
&= \frac{1}{5}(a_i + a_j)\left(\sum_{l\in\{1,\ldots,5\}\setminus\{i\}} p_i^l + \sum_{l\in\{1,\ldots,5\}\setminus\{j\}} p_j^l\right) \\
&= \frac{1}{5}\left(\sum_{l\in\{1,\ldots,5\}\setminus\{i,j\}} (a_i + a_j)(p_i^l + p_j^l)\right) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&\leq \frac{1}{15}\left(\sum_{l\in\{1,\ldots,5\}\setminus\{i,j\}} (\alpha - a_l)\right) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&= \frac{1}{15}(2\alpha + a_i + a_j) + \frac{1}{5}(a_i + a_j)(p_i^j + p_j^i) \\
&= \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}(p_i^j + p_j^i)\right).
\end{aligned}$$

We need to show that $f_{ij}(p_i,p_j) \leq \frac{4}{15}\alpha$ for $1 \leq i \neq j \leq 5$.

If $a_i > (\alpha - a_j)/2$ and $a_j > (\alpha - a_i)/2$, then $p_i^j = p_j^i = 0$, and hence $f_{ij}(p_i,p_j) \leq \frac{3}{15}\alpha < \frac{4}{15}\alpha$.

Now assume $a_i > (\alpha - a_j)/2$ and $a_j \leq (\alpha - a_i)/2$. Then $p_i^j = 0$ and $p_j^i \leq \max\{1/2 - a_j/(\alpha - a_i), 1/3\}$. Suppose $1/2 - a_j/(\alpha - a_i) > 1/3$. Then $a_j < (\alpha - a_i)/6$; and hence, since

$a_i > (\alpha - a_j)/2$, we have $a_i > (\alpha - \alpha/6 + a_i/6)/2$. Solving this inequality for a_i, we have $a_i > 5\alpha/11$ which is in conflict with our assumption. Therefore, $1/2 - a_j/(\alpha - a_i) \le 1/3$, and so $p_j^i \le 1/3$. Hence

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\frac{1}{3}\right) \le \frac{4}{15}\alpha.$$

By symmetry, if $a_j > (\alpha - a_i)/2$ and $a_i \le (\alpha - a_j)/2$, then $f_{ij}(p_i, p_j) \le \frac{4}{15}\alpha$.

So we are left with the case when $a_i \le (\alpha - a_j)/2$ and $a_j \le (\alpha - a_i)/2$. Then $a_i + a_j \le \alpha - (a_i + a_j)/2$, and so $a_i + a_j \le 2\alpha/3$. Moreover, $p_i^j \le \max\{1/2 - a_i/(\alpha - a_j), 1/3\}$ and $p_j^i \le \max\{1/2 - a_j/(\alpha - a_i), 1/3\}$.

If $1/2 - a_i/(\alpha - a_j) > 1/3$ and $1/2 - a_j/(\alpha - a_i) > 1/3$, then $6a_i + a_j < \alpha$ and $6a_j + a_i < \alpha$. Hence $a_i + a_j < 2\alpha/7$, and so (since $p_i^j \le 1/2$ and $p_j^i \le 1/2$),

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{2} + \frac{1}{2}\right)\right) < \frac{2}{15}\alpha + \frac{2}{7}\frac{4}{15}\alpha < \frac{4}{15}\alpha.$$

If $1/2 - a_i/(\alpha - a_j) > 1/3$ and $1/2 - a_j/(\alpha - a_i) \le 1/3$, then $6a_i + a_j \le \alpha$ and $p_j^i \le 1/3$. Since $a_j \le (\alpha - a_i)/2$, $a_i + 2a_j \le \alpha$. So $11(a_i + a_j) = 6a_i + a_j + 5(a_i + 2a_j) \le 6\alpha$, and hence $a_i + a_j \le 6\alpha/11$. Then

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{2} + \frac{1}{3}\right)\right) \le \frac{2}{15}\alpha + \frac{6}{11}\frac{7}{30}\alpha < \frac{4}{15}\alpha.$$

The case when $1/2 - a_i/(\alpha - a_j) \le 1/3$ and $1/2 - a_j/(\alpha - a_i) > 1/3$ is symmetric.

Therefore, we may assume that $1/2 - a_i/(\alpha - a_j) \le 1/3$ and $1/2 - a_j/(\alpha - a_i) \le 1/3$. Then $p_i^j \le 1/3$ and $p_j^i \le 1/3$. Recall that $a_i + a_j \le 2\alpha/3$. Hence

$$f_{ij}(p_i, p_j) \le \frac{2}{15}\alpha + (a_i + a_j)\left(\frac{1}{15} + \frac{1}{5}\left(\frac{1}{3} + \frac{1}{3}\right)\right) \le \frac{2}{15}\alpha + \frac{2}{3}\frac{1}{5}\alpha = \frac{4}{15}\alpha.$$

∎

Using the same proof of Theorem 2.5, with Lemma 2.2 and Lemma 4.1 in place of Lemma 2.4, we have the following results on 4-partitions and 5-partitions.

Theorem 4.2 $f(4, m) \le m/3 + O(m^{4/5})$.

Theorem 4.3 $f(5, m) \le 4m/15 + O(m^{4/5})$.

Recall that the graphs $K_{1,n}$ give $f(4, m) \ge m/3$ and $f(5, m) \ge m/4$.

When $k = 4$, $12m/((k+2)(k+1)) = 3/5 > 1/3$. So as a consequence of Theorem 4.2, Conjecture 1.2 holds for $k = 4$. When $k = 5$, $12m/((k+2)(k+1)) = 2/7 > 4/15$. Hence, Theorem 4.3 establishes Conjecture 1.2 for $k = 5$.

5 Simultaneous bounds for 3-partitions and 4-partitions

In this section, we study the following problem suggested by Bollobás and Scott [7].

Problem 5.1 *For any integer $k \geq 2$ and for any graph G with m edges and n vertices, is it possible to find a partition $V_1, \ldots, V_k$ of $V(G)$ such that for $1 \leq i \leq k$,*

$$e(V_i) \leq \frac{m}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2m+\frac{1}{4}} - \frac{1}{2}\right),$$

and for $1 \leq i \neq j \leq k$,

$$e(V_i \cup V_j) \leq \frac{12m}{(k+2)(k+1)} + O(n)?$$

Recall that Bollobás and Scott [5] showed the existence of a k-partition satisfying the above bound on $e(V_i)$, and K_{kn+1} are the only extremal graphs. Also recall that the bound on $e(V_i \cup V_j)$ is best possible for K_{k+2}.

We show that for $k = 3$ and $k = 4$, one can find partitions that satisfy these bounds asymptotically. For large k, a similar approach as in the proofs of Lemma 2.4 and Theorem 2.5 may be used to give some bounds.

Note that in the proofs to follow, we will use the fact that the maximum of $x(a-x)$, $a > 0$, is $a^2/4$.

Lemma 5.2 *Let $a_j \geq 0$ for $j = 1, 2, 3$ such that $\alpha := a_1 + a_2 + a_3 > 0$, let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \leq i \neq j \leq 3$, and let $g_i(x_i) = a_i x_i$ for $1 \leq i \leq 3$. Then there exist $p_i \in [0, 2/3]$, $1 \leq i \leq 3$, such that $\sum_{i=1}^{3} p_i = 1$, $f_{ij}(p_i, p_j) \leq 5\alpha/9$ for $1 \leq i \neq j \leq 3$, and $g_i(p_i) \leq \alpha/9$ for $1 \leq i \leq 3$.*

Proof. First, assume that $a_i < 2\alpha/3$ for all $i = 1, 2, 3$. Let $p_i = 2/3 - a_i/\alpha$. Then $p_i \in [0, 2/3]$, $i = 1, 2, 3$, and $p_1 + p_2 + p_3 = 1$. Moreover, for $1 \leq i \neq j \leq 3$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{\alpha}\left(\frac{4}{3} - \frac{a_i + a_j}{\alpha}\right)\alpha \leq \frac{4}{9}\alpha < \frac{5}{9}\alpha$$

and, for $i = 1, 2, 3$,

$$g_i(p_i) = \frac{a_i}{\alpha}\left(\frac{2}{3} - \frac{a_i}{\alpha}\right)\alpha \leq \frac{1}{9}\alpha.$$

Next assume that some $a_i > 5\alpha/6$, say $a_3 > 5\alpha/6$. So $a_1 + a_2 \leq \alpha/6$. We choose $p_1 = p_2 = 4/9$ and $p_3 = 1/9$. Then $f_{12}(p_1, p_2) < \alpha/6 < 5\alpha/9$; $f_{i3}(p_i, p_3) \leq 5\alpha/9$ for $i = 1, 2$; $g_3(p_3) \leq \alpha/9$; and $g_i(p_i) \leq (\alpha/6)(4/9) = 2\alpha/27 < \alpha/9$ for $i = 1, 2$.

Therefore, we may assume that there exists some a_i, say a_3, such that $2\alpha/3 \leq a_3 \leq 5\alpha/6$. Then $\alpha/6 \leq a_1 + a_2 \leq \alpha/3$. Let $p_3 = 0$ and $p_i = 2/3 - a_i/(3(a_1 + a_2))$ for $i = 1, 2$. Then $p_i \in [0, 2/3]$ and $p_1 + p_2 + p_3 = 1$.

Clearly, $g_3(p_3) = 0$ and, for $i = 1, 2$,

$$g_i(p_i) = \frac{a_i}{3(a_1 + a_2)}\left(\frac{2}{3} - \frac{a_i}{3(a_1 + a_2)}\right)3(a_1 + a_2) \leq \frac{3}{9}(a_1 + a_2) \leq \frac{1}{9}\alpha.$$

Note that $f_{12}(p_1, p_2) \le (a_1 + a_2) \le \alpha/3 < 5\alpha/9$. So it remains to show that $f_{13}(p_1, p_3) \le 5\alpha/9$ and $f_{23}(p_2, p_3) \le 5\alpha/9$. By symmetry we only need to prove $f_{13}(p_1, p_3) \le 5\alpha/9$.

Note that $f_{13}(p_1, p_3) = (a_1 + a_3)(2/3 - a_1/(3(\alpha - a_3)))$, which may be viewed as a function of a_1, a_3 (while fixing α). We look for the maximal value of $h(a_1, a_3) := f_{13}(p_1, p_3)$ subject to $2\alpha/3 \le a_1 + a_3 \le \alpha$ and $2\alpha/3 \le a_3 \le 5\alpha/6$. Taking partial derivatives and setting them to 0, we have

$$\frac{\partial h}{\partial a_1} = \frac{2}{3} - \frac{a_1}{3(\alpha - a_3)} - \frac{a_1 + a_3}{3(\alpha - a_3)} = 0,$$

and

$$\frac{\partial h}{\partial a_3} = \frac{2}{3} - \frac{a_1}{3(\alpha - a_3)} - \frac{1}{3} a_1 \frac{a_1 + a_3}{(\alpha - a_3)^2} = 0.$$

Then $a_1/(\alpha - a_3) = 1$ (from $\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_3}$), and hence $a_3 = 0$ (from $\frac{\partial h}{\partial a_1} = 0$), a contradiction. So the maximal value of h occurs on the boundary of the region defined by $2\alpha/3 \le a_1 + a_3 \le \alpha$ and $2\alpha/3 \le a_3 \le 5\alpha/6$.

When $a_1 + a_3 = 2\alpha/3$, then $a_1 = 0$ and $a_3 = 2\alpha/3$, and hence $h = 4\alpha/9$. When $a_1 + a_3 = \alpha$ then $h = \alpha/3$. When $a_3 = 2\alpha/3$ then $h = (a_1 + 2\alpha/3)(2/3 - a_1/\alpha) = (2/3 + a_1/\alpha)(2/3 - a_1/\alpha)\alpha \le 4\alpha/9$. When $a_3 = 5\alpha/6$, then $h \le (a_1 + 5\alpha/6)(2/3 - 2a_1/\alpha) = (5/6 + a_1/\alpha)(2/3 - 2a_1/\alpha)\alpha \le 5\alpha/9$. Hence $f_{13}(p_1, p_3) \le 5\alpha/9$. ∎

The next lemma is for 4-partitions.

Lemma 5.3 *Let $a_j \ge 0$ for $j = 1, 2, 3, 4$ such that $\alpha := a_1 + a_2 + a_3 + a_4 > 0$, let $f_{ij}(x_i, x_j) = (a_i + a_j)(x_i + x_j)$ for $1 \le i \ne j \le 4$, and let $g_i(x_i) = a_i x_i$ for $1 \le i \le 4$. Then there exist $p_i \in [0, 1/2]$, $1 \le i \le 4$, such that $\sum_{i=1}^4 p_i = 1$, $f_{ij}(p_i, p_j) \le 2\alpha/5$ for $1 \le i \ne j \le 4$, and $g_i(p_i) \le \alpha/16$ for $1 \le i \le 4$.*

Proof. First, suppose $a_i < \alpha/2$ for all $1 \le i \le 4$. Let $p_i = 1/2 - a_i/\alpha$. Then $p_i \in [0, 1/2]$ for $1 \le i \le 4$, and $\sum_{i=1}^4 p_i = 1$. Moreover, for $1 \le i \ne j \le 4$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{\alpha}\left(1 - \frac{a_i + a_j}{\alpha}\right)\alpha \le \frac{1}{4}\alpha < \frac{2}{5}\alpha,$$

and for $1 \le i \le 4$,

$$g_i(p_i) = \frac{a_i}{\alpha}\left(\frac{1}{2} - \frac{a_i}{\alpha}\right)\alpha \le \frac{1}{16}\alpha.$$

Now assume that some $a_i > 4\alpha/5$, say $a_4 > 4\alpha/5$. Then $a_1 + a_2 + a_3 \le \alpha/5$. Let $p_1 = p_2 = p_3 = 5/16$ and $p_4 = 1/16$. Then for $i = 1, 2, 3$, $f_{i4}(p_i, p_4) \le 6\alpha/16 < 2\alpha/5$; for $1 \le i \ne j \le 3$, $f_{ij}(p_i, p_j) \le \alpha/5 < 2\alpha/5$; $g_4(p_4) \le \alpha/16$; and for $i = 1, 2, 3$, $g_i(p_i) \le (\alpha/5)(5/16) = \alpha/16$.

So we may assume that there exists some a_i, say a_4, such that $\alpha/2 \le a_4 \le 4\alpha/5$. Then $\alpha/5 \le a_1 + a_2 + a_3 \le \alpha/2$. Let $p_4 = 0$ and $p_i = 1/2 - a_i/(2(\alpha - a_4))$. Then $p_i \in [0, 1/2]$ and $\sum_{i=1}^4 p_i = 1$.

Clearly, $g_4(p_4) = 0$. Note that $\alpha - a_4 \le \alpha/2$. So for $i = 1, 2, 3$

$$g_i(p_i) = \frac{a_i}{2(\alpha - a_4)}\left(\frac{1}{2} - \frac{a_i}{2(\alpha - a_4)}\right)2(\alpha - a_4) \le \frac{1}{16}\alpha;$$

and for $1 \le i \neq j \le 3$,

$$f_{ij}(p_i, p_j) = \frac{a_i + a_j}{2(\alpha - a_4)} \left(1 - \frac{a_i + a_j}{2(\alpha - a_4)}\right) 2(\alpha - a_4) \le \frac{1}{4}\alpha < \frac{2}{5}\alpha.$$

Thus it remains to prove $f_{i4}(p_i, p_4) \le 2\alpha/5$ for $i = 1, 2, 3$. By symmetry, we only prove $f_{14}(p_1, p_4) \le 2\alpha/5$. Note that $h(a_1, a_4) := f_{14}(p_1, p_4) = (a_1 + a_4)(1/2 - a_1/(2(\alpha - a_4)))$ may be viewed as a function of a_1, a_4 (while fixing α), and we look for its maximal value subject to $\alpha/2 \le a_1 + a_4 \le \alpha$ and $\alpha/2 \le a_4 \le 4\alpha/5$.

Taking partial derivatives and setting them to 0, we have

$$\frac{\partial h}{\partial a_1} = \frac{1}{2} - \frac{a_1}{2(\alpha - a_4)} - \frac{1}{2}\frac{a_1 + a_4}{\alpha - a_4} = 0,$$

and

$$\frac{\partial h}{\partial a_4} = \frac{1}{2} - \frac{a_1}{2(\alpha - a_4)} - \frac{1}{2} a_1 \frac{a_1 + a_4}{(\alpha - a_4)^2} = 0.$$

Then $a_1/(\alpha - a_4) = 1$ (from $\frac{\partial h}{\partial a_1} = \frac{\partial h}{\partial a_4}$), and so $a_4 < 0$ (from $\frac{\partial h}{\partial a_1} = 0$), a contradiction. Thus, the maximal value of h occurs when $a_1 + a_4 \in \{\alpha/2, \alpha\}$ or $a_4 \in \{\alpha/2, 4\alpha/5\}$.

When $a_1 + a_4 = \alpha/2$, we have $a_1 = 0$ and $a_4 = \alpha/2$, and hence $h = \alpha/4$. When $a_1 + a_4 = \alpha$, then $h = 0$. When $a_4 = \alpha/2$ then $h = \alpha(1/2 + a_1/\alpha)(1/2 - a_1/\alpha) \le \alpha/4$. When $a_4 = 4\alpha/5$, then $h = \alpha(4/5 + a_1/\alpha)(1/2 - 5a_1/(2\alpha)) \le 2\alpha/5$. Hence $f_{14}(a_1, a_4) \le 2\alpha/5$. ∎

Now we use Lemma 5.2 and (essentially) the same proof of Theorem 2.5 to prove the following.

Theorem 5.4 *Let G be a graph with m edges. Then there is a partition X_1, X_2, X_3 of $V(G)$ such that for $1 \le i \le 3$,*

$$e(X_i) \le \frac{1}{9}m + O(m^{4/5}),$$

and for $1 \le i \neq j \le 3$,

$$e(X_i \cup X_j) \le \frac{5}{9}m + O(m^{4/5}).$$

Proof. We may assume that G is connected. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_1) \ge d(v_2) \ge \ldots \ge d(v_n)$. Let $V_1 = \{v_1, \ldots, v_t\}$ with $t = \lfloor m^\alpha \rfloor$, where $0 < \alpha < 1/2$. Then $t < n$, $e(V_1) \le \frac{1}{2}m^{2\alpha}$, and $d(v_{t+1}) \le 2m^{1-\alpha}$. Let $V_2 := V(G) \setminus V_1 = \{u_1, \ldots, u_{n-t}\}$ such that $e(u_i, V_1 \cup \{u_1, \ldots, u_{i-1}\}) > 0$ for $i = 1, \ldots, n - t$.

Fix a random 3-partition $V_1 = Y_1 \cup Y_2 \cup Y_3$, and assign each member of Y_i the color i, $1 \le i \le 3$. Extend this coloring to $V(G)$ such that each vertex $u_i \in V_2$ is independently assigned the color j with probability p_j^i, where $\sum_{j=1}^3 p_j^i = 1$ and p_j^i will be determined later. Let Z_i denote the indicator random variable of the event of coloring u_i.

Let $G_i = G[V_1 \cup \{u_1, \cdots, u_i\}]$ for $i = 1, \ldots, n - t$, and let $G_0 = G[V_1]$. Let $X_j^0 = Y_j$ and $x_{jl}^0 = e(X_j^0 \cup X_l^0)$ for $1 \le j, l \le 3$. For $i = 1, \ldots, n - t$ and $1 \le j, l \le 3$, define

$$\begin{aligned}
X_j^i &:= \{\text{vertices of } G_i \text{ with color } j\}, \\
x_{jl}^i &:= e(X_j^i \cup X_l^i), \\
\Delta x_{jl}^i &:= x_{jl}^i - x_{jl}^{i-1}, \\
b_j^i &:= e(u_i, X_j^{i-1}).
\end{aligned}$$

When $j = l$, let $x_j^i := x_{jl}^i$ and $\Delta x_j^i = \Delta x_{jl}^i$. Note that b_j^i depends on $(Z_1, \ldots, Z_{i-1})$ only and $\sum_{j=1}^3 b_j^i = e(u_i, G_{i-1})$ is in dependent of $(Z_1, \ldots, Z_{i-1})$. Let $a_j^i = \sum_{(Z_1,\ldots,Z_{i-1})} P(Z_1, \ldots, Z_{i-1}) b_j^i$, which is determined by p_j^s, $1 \le j \le 3$ and $1 \le s \le i-1$. As in the proof of Theorem 2.5, for $1 \le i \le n-t$ and $1 \le j \ne l \le 3$ we have

$$\mathbb{E}(\Delta x_{jl}^i) = (a_j^i + a_l^i)(p_j^i + p_l^i),$$

and for $1 \le i \le n-t$ we have

$$\mathbb{E}(\Delta x_j^i) = a_j^i p_j^i.$$

By Lemma 5.2, there exist $p_j^i \in [0, 2/3]$, $1 \le j \le 3$, such that $\sum_{j=1}^3 p_j^i = 1$; for $1 \le i \le n-t$ and $1 \le j \ne l \le 3$,

$$\mathbb{E}(\Delta x_{jl}^i) \le \frac{5}{9} \sum_{j=1}^3 a_j^i = \frac{5}{9} \sum_{j=1}^3 b_j^i = \frac{5}{9} e(u_i, G_{i-1});$$

and for $1 \le i \le n-t$,

$$\mathbb{E}(\Delta x_j^i) \le \frac{1}{9} \sum_{j=1}^3 a_j^i = \frac{1}{9} \sum_{j=1}^3 b_j^i = \frac{1}{9} e(u_i, G_{i-1}).$$

Note that p_j^i is determined by a_j^i, $1 \le j \le 3$; and hence p_j^i is recursively defined by p_j^s, $1 \le j \le 3$ and $1 \le s \le i-1$. Now

$$\mathbb{E}(x_{jl}^{n-t}) = \frac{5}{9} \sum_{i=1}^{n-t} e(u_i, G_{i-1}) + x_{jl}^0 \le \frac{5}{9} m + e(V_1),$$

and

$$\mathbb{E}(x_j^{n-t}) \le \frac{1}{9} \sum_{i=1}^{n-t} e(u_i, G_{i-1}) + x_j^0 \le \frac{1}{9} m + e(V_1).$$

Clearly, changing the color of u_i (i.e., changing Z_i) affects $x_{jl} := x_{jl}^{n-t}$ and $x_j := x_j^{n-t}$ by at most $d(u_i)$. So by Lemma 2.1,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) \le \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right),$$

and

$$\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) \le \exp\left(-\frac{z^2}{8m^{2-\alpha}}\right).$$

Let $z = (8 \ln 6)^{\frac{1}{2}} m^{1-\frac{\alpha}{2}}$. Then for $1 \le j \ne l \le 3$,

$$\mathbb{P}\left(x_{jl} > \mathbb{E}(x_{jl}) + z\right) < \frac{1}{6},$$

and for $1 \le j \le 3$,

$$\mathbb{P}\left(x_j > \mathbb{E}(x_j) + z\right) < \frac{1}{6}.$$

So there exists a partition $V(G) = X_1 \cup X_2 \cup X_3$ such that for $1 \le j \ne l \le 3$,

$$e(X_j \cup X_l) \le \mathbb{E}(x_{jl}) + z \le \frac{5}{9}m + o(m),$$

and for $1 \le j \le 3$,

$$e(X_j) \le \mathbb{E}(x_j) + z \le \frac{1}{9}m + o(m).$$

The $o(m)$ term in both expressions is

$$\frac{1}{2}m^{2\alpha} + (8\ln 6)^{\frac{1}{2}}\, m^{1-\frac{\alpha}{2}}.$$

Picking $\alpha = \frac{2}{5}$ to minimize $\max\{2\alpha, 1-\alpha/2\}$, the $o(m)$ term becomes $O(m^{\frac{4}{5}})$. ∎

By the same argument as in the proof of Theorem 5.4, using Lemma 5.3 instead of Lemma 5.2, we have the following result.

Theorem 5.5 *Let G be a graph with m edges. Then there is a partition X_1, X_2, X_3, X_4 of $V(G)$ such that for $1 \le i \le 4$,*

$$e(X_i) \le \frac{1}{16}m + O(m^{4/5}),$$

and for $1 \le i \ne j \le 4$,

$$e(X_i \cup X_j) \le \frac{2}{5}m + O(m^{4/5}).$$

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